

Multipoint Normal Differential Operators of Second Order

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Abstract

In this work it is described all normal extensions of a multipoint minimal operator generated by linear multipoint differential-operator expression for second order in the Hilbert space of vector functions in terms of boundary values at the endpoints of the infinitely many subintervals. Finally, a spectrum structure of such extensions has been investigated.

Keywords: Direct sum of Hilbert spaces and operators; multipoint selfadjoint, formally normal and normal operators; extension.

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1 Introduction

It is known that traditional infinite direct sum of Hilbert spaces H_n , $n \geq 1$ and infinite direct sum of operators A_n in H_n , $n \geq 1$ are define as

$$H = \bigoplus_{n=1}^{\infty} H_n = \left\{ u = (u_n) : u_n \in H_n, n \geq 1 \text{ and } \sum_{n=1}^{\infty} \|u_n\|_{H_n}^2 < +\infty \right\},$$

$$A = \bigoplus_{n=1}^{\infty} A_n, D(A) = \{ u = (u_n) \in H : u_n \in D(A_n), n \geq 1 \text{ and}$$

$$Au = (A_n u_n) \in H \}.$$

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A linear space H is a Hilbert space with norm corresponding to inner product

$$(u, v)_H = \sum_{n=1}^{\infty} (u_n, v_n)_{H_n}, \quad u, v \in H[1].$$

The general theory of linear closed operators in Hilbert spaces and its applications to physical problems have been investigated by many mathematicians (for example, see [1]).

However, many physical problems support to study a theory of linear operators in direct sums in Hilbert spaces (for example, see [2]-[6] and references in them)today.

Notice that a detail analysis of normal subspaces and operators in Hilbert spaces has been studied in [7] (see references in it).

In this work in first section a connection between multipoint and two-points normal operators are investigated.

In second section all normal extensions of multipoint formally normal operators are described in terms of boundary values in the endpoints of the infinitely many subintervals. Furthermore, a spectrum structure has been researched of such extensions.

2 The Minimal and Maximal Operators

Along of this work (a_n) and (b_n) will be sequences of real numbers such that

$$-\infty < a_n < b_n \leq a_{n+1} < \cdots < +\infty,$$

H_n is any Hilbert space, $\Delta_n = (a_n, b_n)$, $L_n^2 = L^2(H_n, \Delta_n)$, $L^2 = \bigoplus_{n=1}^{\infty} L^2(H_n, \Delta_n)$,

$(\cdot, \cdot)_{H_n} = (\cdot, \cdot)_n$, $n \geq 1$, $W_2^2 = \bigoplus_{n=1}^{\infty} W_2^2(H_n, \Delta_n)$, $W_2^2 = \bigoplus_{n=1}^{\infty} W_2^2(H_n, \Delta_n)$, $H = \bigoplus_{n=1}^{\infty} H_n$, $cl(T)$ -closure of the operator T , E is an identity operator in corresponding spaces. $l(\cdot)$ is a linear multipoint differential-operator expression for second order in L^2 in the following form

$$l(u) = (l_n(u_n)) \tag{1}$$

and for each $n \geq 1$

$$l_n(u_n) = -u_n'' + iA_n u_n, \tag{2}$$

where $A_n : D(A_n) \subset H_n \rightarrow H_n$ is a linear positive defined selfadjoint operator in H_n .

It is clear that formally adjoint expression to (2) in the Hilbert space L_n^2 is in the form

$$l_n^+(v_n) = -v_n'' - iA_n^*v_n, \quad n \geq 1. \quad (3)$$

Define an operator L'_{n0} on the dense manifold of vector functions D'_{n0} in L_n^2 ,

$$D'_{n0} := \{u_n \in L_n^2 : u_n = \sum_{k=1}^m \phi_k f_k, \phi_k \in C_0^\infty(\Delta_n), f_k \in D(A_n), \\ k = 1, 2, \dots, m; m \in \mathbb{N}\}$$

as $L'_{n0}u_n := l_n(u_n)$, $n \geq 1$.

Since the operator $A_n > 0$, then from the relation

$$Im(L'_{n0}u_n, u_n)_{L_n^2} = (A_n u_n, u_n)_{L_n^2} \geq 0, \quad u_n \in D'_{n0}, \quad n \geq 1$$

implies that L'_{n0} is a dissipative in L_n^2 , $n \geq 1$. Hence the operator L'_{n0} has a closure in L_n^2 , $n \geq 1$. The closure $cl(L'_{n0})$ of the operator L'_{n0} is called the minimal operator generated by differential-operator expression (2) and it is denoted by L_{n0} in L_n^2 , $n \geq 1$. The operator L_0 defined by

$$D(L_0) := \left\{ u = (u_n) : u_n \in D(L_{n0}), \quad n \geq 1, \quad \sum_{n=1}^{\infty} \|L_{n0}u_n\|_{L_n^2}^2 < +\infty \right\},$$

$$L_0 u := (L_{n0}u_n), \quad u \in D(L_0), \quad L_0 : D(L_0) \subset L^2 \rightarrow L^2$$

is called a minimal operator (multipoint) generated by differential-operator expression (1) in Hilbert space L^2 and denoted by $L_0 = \bigoplus_{n=1}^{\infty} L_{n0}$.

In a similar way the minimal operator (twopoints) L_{n0}^+ in L_n^2 , $n \geq 1$ for the formally adjoint linear differential-operator expression (3) can be constructed.

In this case the operator L_0^+ defined by

$$D(L_0^+) := \left\{ v := (v_n) : v_n \in D(L_{n0}^+), \quad n \geq 1, \quad \sum_{n=1}^{\infty} \|L_{n0}^+v_n\|_{L_n^2}^2 < +\infty \right\},$$

$L_0^+ v := (L_{n0}^+ v_n)$, $v \in D(L_0^+)$, $L_0^+ : D(L_0^+) \subset L^2 \rightarrow L^2$ is called a minimal operator (multipoint) generated by $l^+(v) = (l_n^+(v_n))$ in the Hilbert space L^2 and denoted by $L_0^+ = \bigoplus_{n=1}^{\infty} L_{n0}^+$.

Note that the following proposition is true.

2.1. Theorem. The minimal operators L_0 and L_0^+ are densely defined closed operators in L^2 .

The following defined operators in L^2

$$L := (L_0^+)^* = \bigoplus_{n=1}^{\infty} L_n \text{ and } L^+ := (L_0)^* = \bigoplus_{n=1}^{\infty} L_n^+$$

are called maximal operators (multipoint) for the differential-operator expression $l(\cdot)$ and $l^+(\cdot)$ respectively. It is clear that $Lu = (l_n(u_n))$, $u \in D(L)$,

$$D(L) := \left\{ u = (u_n) \in L^2 : u_n \in D(L_n), n \geq 1, \sum_{n=1}^{\infty} \|L_n u_n\|_{L_n^2}^2 < \infty \right\},$$

$$L^+ v = (l_n^+(v_n)), v \in D(L^+),$$

$$D(L^+) := \left\{ v = (v_n) \in L^2 : v_n \in D(L_n^+), n \geq 1, \sum_{n=1}^{\infty} \|L_n^+ v_n\|_{L_n^2}^2 < \infty \right\}$$

and $L_0 \subset L$, $L_0^+ \subset L^+$.

Furthermore, the validity of following proposition is clear.

2.2. Theorem. The domain of the operators L and L_0 are

$$\begin{aligned} D(L) = \{ & u = (u_n) \in L^2 : (1) \text{ for each } n \geq 1 \text{ vector function } u_n \in L_n^2, \text{ derivative } u'_n \\ & \text{is absolutely continuous in interval } \Delta_n; \\ & (2) l_n(u_n) \in L_n^2, n \geq 1; (3) l(u) = (l_n(u_n)) \in L^2 \} \\ = \{ & u = (u_n) \in L^2 : u_n \in D(L_n), n \geq 1 \text{ and } l(u) = (l_n(u_n)) \in L^2 \}, \end{aligned}$$

$$D(L_0) = \{ u = (u_n) \in D(L) : u_n(a_n) = u_n(b_n) = u'_n(a_n) = u'_n(b_n) = 0, n \geq 1 \}.$$

2.3. Remark. If for any $n \geq 1$, $u_n \in D(L_{n0})$, then $u = (u_n)$ may not be in $D(L_0)$ in general. Indeed, choose the function in form

$$u_n(t) := c_n \sin^2 \left(n\pi \frac{t - a_n}{b_n - a_n} \right) f_n,$$

$c_n \in \mathbb{C}$, $t \in \Delta_n$, $(f_n) \in D(A)$, $f_n \neq 0$, $\alpha_n = \|f_n\|_H$, $n \geq 1$.

In this case it is easily to see that $u_n \in D(L_{n0})$, $n \geq 1$. On the other hand the simple calculations shows that

$$\begin{aligned} \|u\|_{L^2}^2 &= \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \|u_n\|_{L_n^2}^2 dt = \sum_{n=1}^{\infty} \alpha_n^2 c_n^2 \int_{a_n}^{b_n} \sin^4 \left(n\pi \frac{t - a_n}{b_n - a_n} \right) dt \\ &= \frac{3}{4} \sum_{n=1}^{\infty} \alpha_n^2 c_n^2 (b_n - a_n). \end{aligned}$$

Here, if put $\alpha_n := \frac{1}{n}$, $c_n := \left[\frac{4}{3(b_n - a_n)} \right]^{1/2} \cdot \frac{1}{\alpha_n}$, $n \geq 1$,

then from the last relation implies that $\|u\|_{L^2}^2 = \sum 1 = +\infty$, i.e $u \notin L^2$.

2.4. Remark. If $A_n \in B(H)$, $n \geq 1$ and $\sup_{n \geq 1} \|A_n\| \leq c < +\infty$, then for any $u = (u_n) \in L^2$ we have $(Au) = (A_n u_n) \in L^2$.

Now the following results can be proved .

2.5. Theorem. If a minimal operator L_0 is formally normal in L^2 , then $D(L_0) \subset W_2^2$ and $AD(L_0) \subset L^2$.

2.6. Theorem If $A^{1/2}W_2^2 \subset W_2^2$, then minimal operator L_0 is formally normal in L^2 .

Proof: In this case from the following relations

$$L_0^+ u = -u'' - iAu = (-u'' + iAu) - 2iAu = L_0 u - 2iAu, \quad u \in D(L_0),$$

$$L_0 u = -u'' + iAu = (-u'' - iAu) + 2iAv = L_0^+ u + 2iAu, \quad u \in D(L_0^+),$$

implies that $D(L_0) = D(L_0^+)$. Since $D(L_0^+) \subset D(L_0^*) = D(L^+)$, it is obtained that $D(L_0) \subset D(L^+)$.

On the other hand for any $u \in D(L_0)$

$$\begin{aligned} \|L_0 u\|_{L^2}^2 &= (-u'' + iAu, -u'' + iAu)_{L^2} = \|u''\|_{L^2}^2 + i[(u'', Au)_{L^2} - (Au, u'')]_{L^2} + \|Au\|_{L^2}^2 \\ &= \|u''\|_{L^2}^2 + \|Au\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} \|L^+ u\|_{L^2}^2 &= (-u'' - iAu, -u'' - iAu)_{L^2} = \|u''\|_{L^2}^2 - i[(u'', Au)_{L^2} - (Au, u'')]_{L^2} + \|Au\|_{L^2}^2 \\ &= \|u''\|_{L^2}^2 + \|Au\|_{L^2}^2. \end{aligned}$$

From this it is established that operator L_0 is formally normal in L^2 .

2.7 Remark. If $A_n \in B(H)$, $n \geq 1$ and $\sup_{n \geq 1} \|A_n\| \leq c < +\infty$, then

$D(L_0) = D(L_0^+)$ and $D(L) = D(L^+)$.

2.8 Remark. If $AW_2^2 \subset L^2$, then $D(L_0) = D(L_0^+)$ and $D(L) = D(L^+)$.

3 Description of Normal Extensions of the Minimal Operator

In this section the main purpose is to describe all normal extensions of the minimal operator L_0 in L^2 in terms in the boundary values of the endpoints of the subintervals .

In first will be shown that there exists normal extension of the minimal operator L_0 . Consider the following extension of the minimal operator L_0

$$\left\{ \begin{array}{l} \tilde{L}u := -u'' + iAu, \quad AW_2^2 \subset W_2^2, \\ D(\tilde{L}) = \{u = (u_n) \in W_2^2 : u_n(a_n) = u_n(b_n), \quad u'_n(a_n) = u'_n(b_n), \quad n \geq 1\}. \end{array} \right.$$

Under the condition to the coefficient A we have

$$\begin{aligned} (\tilde{L}u, v)_{L^2} &= (-u'', v)_{L^2} + i(Au, v)_{L^2} \\ &= (-u', v)'_{L^2} + (u, v')'_{L^2} - (u, v'')_{L^2} + (u, -iAv)_{L^2} \\ &= \sum_{n=1}^{\infty} [(u'_n(b_n), v_n(b_n) - v_n(a_n))_n + (u_n(a_n), v'_n(b_n) - v'_n(a_n))_n] + \\ &\quad + (u, -v'' - iAv)_{L^2} \end{aligned}$$

From this it is obtained

$$\left\{ \begin{array}{l} \tilde{L}^*v := -v'' - iAv, \quad AW_2^2 \subset W_2^2, \\ D(\tilde{L}^*) = \{v = (v_n) \in W_2^2 : v_n(a_n) = v_n(b_n), \quad v'_n(a_n) = v'_n(b_n), \quad n \geq 1\}. \end{array} \right.$$

In this case it is clear that $D(\tilde{L}) = D(\tilde{L}^*)$. On the other hand since for each $u \in D(\tilde{L})$

$$\begin{aligned} \|\tilde{L}u\|_{L^2}^2 &= \|u''\|_{L^2}^2 + i[(u'', Au)_{L^2} - (Au, u'')]_{L^2} + \|Au\|_{L^2}^2, \\ \|\tilde{L}^*u\|_{L^2}^2 &= \|u''\|_{L^2}^2 - i[(u'', Au)_{L^2} - (Au, u'')]_{L^2} + \|Au\|_{L^2}^2 \end{aligned}$$

and

$$\begin{aligned} (u'', Au)_{L^2} - (Au, u'')_{L^2} &= (u', Au)'_{L^2} - (u, Au')'_{L^2} \\ &= \sum_{n=1}^{\infty} [(u'_n(a_n), A_n(u_n(b_n) - u_n(a_n)))_n - (u_n(a_n), A_n(u'_n(b_n) - u'_n(a_n)))_n] = 0, \end{aligned}$$

then $\|\tilde{L}u\|_{L^2} = \|\tilde{L}^*u\|_{L^2}$ for every $u \in D(\tilde{L})$. Consequently, \tilde{L} is a normal extension of the minimal operator L_0 .

The following result established relationship between normal extensions of L_0 and normal extensions of L_{n0} , $n \geq 1$.

3.1 Theorem. If \tilde{L} is a normal extension of the minimal operator L_0 in L^2 , then for any $n \geq 1$,

$$D(\tilde{L}_n) = P_n D(\tilde{L}), \quad \tilde{L}_n u_n = L_n u_n,$$

where $P_n : L^2 \rightarrow L_n^2$ is an orthogonal projection, is a normal extension of the minimal operator L_{n0} in L_n^2 , $n \geq 1$.

Proof: Indeed, in this case firstly it is clear that

$$D(L_{n0}) \subset P_n D(\tilde{L}) \subset D(L_n), \quad n \geq 1 \quad \text{and} \quad D(L_{n0}^+) \subset P_n D(\tilde{L}^*) \subset D(L_n^+), \quad n \geq 1. \quad \text{Now define}$$

$$\tilde{L}_n u_n := l_n(u_n), \quad D(\tilde{L}_n) = P_n D(\tilde{L}), \quad n \geq 1$$

which is an extension of the minimal operator L_{n0} in L_n^2 , $n \geq 1$.

Prove that an extension \tilde{L}_n is a normal in L_n^2 , $n \geq 1$. First of all, note that from above witting relations and normality of \tilde{L} it is implies that $D(\tilde{L}_n) = D(\tilde{L}_n^*)$.

Indeed, now put any $u_n \in D(\tilde{L}_n)$, $n \geq 1$, then $u^* = \{0, 0, \dots, u_n, 0, \dots\} \in D(\tilde{L})$. Because $P_n u^* \in D(\tilde{L}_n)$. On the other hand, since $D(\tilde{L}) = D(\tilde{L}^*)$, then $u^* \in D(\tilde{L}^*)$. Hence, $u_n \in D(\tilde{L}_n^*)$, $n \geq 1$. From this, it is clear that $D(\tilde{L}_n) \subset D(\tilde{L}_n^*)$.

A similar way it is shown that $D(\tilde{L}_n^*) \subset D(\tilde{L}_n)$. Finally we have $D(\tilde{L}_n) = D(\tilde{L}_n^*)$.

On the other hand for any $u_n \in D(\tilde{L}_n)$, $n \geq 1$ the function $u^* \in D(\tilde{L})$.

Since $\|\tilde{L}u^*\|_{L^2} = \|\tilde{L}^*u^*\|_{L^2}$, then

$$\left\| \widetilde{L}_n u_n \right\|_{L_n^2} = \left\| \widetilde{L}_n^* u_n \right\|_{L_n^2}, \quad u_n \in D(\widetilde{L}_n), \quad n \geq 1.$$

Consequently, it is clear that extension \widetilde{L}_n is a normal in L_n^2 , $n \geq 1$.

The minimal operator L_{n0}^r generated by differential expression $l_n^r(u_n) := -u_n''(t)$ in Hilbert space L_n^2 , $n \geq 1$ is symmetric and has equal defect indexes $(\dim H, \dim H)$. Then the minimal operator L_{n0}^r in L_n^2 , $n \geq 1$ has a space of boundary values $(\mathfrak{H}_n, \gamma_1^{(n)}, \gamma_2^{(n)})$, $n \geq 1$ [8] and one of these spaces is in following form

$$\mathfrak{H}_n = H_n \oplus H_n, \quad \gamma_1^{(n)}(u_n) = \{-u_n(a_n), u_n(b_n)\}, \quad \gamma_2^{(n)}(u_n) = \{u_n'(a_n), u_n'(b_n)\},$$

$u_n \in D(L_n^r)$, where L_n^r is defined a maximal operator generated by differential expression $l_n^r(\cdot) = -d^2/dt^2$ in the space L_n^2 , $n \geq 1$.

Now we can prove the following main result of this section in which is given a description of all normal extension of the minimal operator L_0 in L^2 in terms of boundary values of vector functions at the endpoints of subintervals.

3.2 Theorem. Let $A^{1/2}W_2^2 \subset W_2^2$. If $\widetilde{L} = \bigoplus_{n=1}^{\infty} \widetilde{L}_n$ is a normal extension of the minimal operator L_0 in L^2 , then it is generated by differential-operator expression $l(\cdot) = (l_n(\cdot))$ with boundary conditions

$$(W_n - E) \gamma_1^{(n)}(u_n) + i(W_n + E) \gamma_2^{(n)}(u_n) = 0, \quad u_n \in D(L_n),$$

where, $(\mathfrak{H}_n, \gamma_1^{(n)}, \gamma_2^{(n)})$ is a space of boundary values of the minimal operator

L_{n0}^r in L_n^2 , W_n and $\begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} W_n \begin{pmatrix} A_n^{-1/2} & 0 \\ 0 & A_n^{-1/2} \end{pmatrix}$ are unitary operators in \mathfrak{H}_n , $n \geq 1$. The unitary operator $W = \bigoplus_{n=1}^{\infty} W_n$ in $\mathfrak{H} = \bigoplus_{n=1}^{\infty} \mathfrak{H}_n$ is

determined uniquely by the extension \widetilde{L} , i.e $\widetilde{L} = L_W$.

Proof: In this case by the Theorem 3.1 \widetilde{L}_n is a normal extension of the minimal operator L_{n0} in L_n^2 , $n \geq 1$. On the other hand, it is clear that $cl(\text{Re} \widetilde{L}_n)$ is a some selfadjoint extension of the minimal operator L_{n0}^r generated by the differential expression $l_n^r(\cdot) = -d^2/dt^2$ in the L_n^2 , $n \geq 1$. In this case there exists space of boundary values $(\mathfrak{H}_n, \gamma_1^{(n)}, \gamma_2^{(n)})$ for the L_{n0}^r in L_n^2 , $n \geq 1$ [8]. Therefore, the selfadjoint extension $cl(\text{Re} \widetilde{L}_n)$ in L_n^2 is generated by the differential expression $l_n^r(\cdot) = -d^2/dt^2$ and boundary condition

$$(W_n - E) \gamma_1^{(n)}(u_n) + i(W_n + E) \gamma_2^{(n)}(u_n) = 0, \quad u_n \in D(L_n), \quad (4)$$

where W_n is a unitary operator in \mathfrak{H}_n , $n \geq 1$. Note that the unitary operator W_n is determined uniquely by the extension $cl\left(\widetilde{ReL_n}\right)$, i.e. $cl\left(\widetilde{ReL_n}\right) = L_n(W_n)$, $n \geq 1$ [8], [9].

On the other hand, $cl\left(\widetilde{ImL_n}\right)$ is a selfadjoint operator which is acting in L_n^2 with domain $D\left(cl\left(\widetilde{ReL_n}\right)\right)$, $n \geq 1$. Since selfadjoint operators $cl\left(\widetilde{ReL_n}\right)$ and $cl\left(\widetilde{ImL_n}\right)$ are commutative in the space L_n^2 , $n \geq 1$, then for each $u_n \in D\left(cl\left(\widetilde{ReL_n}\right)\right)$

$$\begin{aligned} & \left(cl\left(\widetilde{ReL_n}\right) u_n, cl\left(\widetilde{ImL_n}\right) u_n \right)_{L_n^2} - \left(cl\left(\widetilde{ImL_n}\right) u_n, cl\left(\widetilde{ReL_n}\right) u_n \right)_{L_n^2} = \\ & = (-u_n'', A_n u_n)_{L_n^2} - (A_n u_n, -u_n'')_{L_n^2} \\ & = \left(\gamma_1^{(n)}\left(A_n^{1/2} u_n\right), \gamma_2^{(n)}\left(A_n^{1/2} u_n\right) \right)_{\mathfrak{H}_n} - \left(\gamma_2^{(n)}\left(A_n^{1/2} u_n\right), \gamma_1^{(n)}\left(A_n^{1/2} u_n\right) \right)_{\mathfrak{H}_n} = 0 \end{aligned}$$

Since for each $n \geq 1$ $A_n^{-1} \in B(H)$, then last equation means that the linear relation

$$\tilde{\theta}_n := \left\{ \left\{ \gamma_1^{(n)}\left(A_n^{1/2} u_n\right), \gamma_2^{(n)}\left(A_n^{1/2} u_n\right) \right\} : u_n \in D\left(cl\left(\widetilde{ReL_n}\right)\right) \right\}$$

is selfadjoint in $\mathfrak{H}_n \oplus \mathfrak{H}_n$ (see [7], [9]). Consequently, there is a uniquely V_n unitary operator in \mathfrak{H}_n , such that

$$(V_n - E) \gamma_1^{(n)}\left(A_n^{1/2} u_n\right) + i(V_n + E) \gamma_2^{(n)}\left(A_n^{1/2} u_n\right) = 0 \quad (5)$$

$u_n \in D\left(cl\left(\widetilde{ReL_n}\right)\right)$, $n \geq 1$ [9].

Now consider the mappings $\gamma_1^{(n)}\left(A_n^{1/2} \cdot\right)$ and $\gamma_2^{(n)}\left(A_n^{1/2} \cdot\right)$ defined in \mathfrak{H}_n , $n \geq 1$.

1. For $u_n \in D\left(cl\left(\widetilde{ReL_n}\right)\right)$, $n \geq 1$, it is clear that

$$\begin{aligned} & \gamma_1^{(n)}\left(A_n^{1/2} u_n\right) = \left\{ -A_n^{1/2} u_n(0), A_n^{1/2} u_n(1) \right\} \\ & = \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} \begin{pmatrix} -u_n(0) \\ u_n(1) \end{pmatrix} = \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} \gamma_1^{(n)}(u_n) \end{aligned}$$

and

$$\begin{aligned}\gamma_2^{(n)}(A_n^{1/2}u_n) &= \left\{ A_n^{1/2}u'_n(0), A_n^{1/2}u'_n(1) \right\} \\ &= \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} \begin{pmatrix} u'_n(0) \\ u'_n(1) \end{pmatrix} = \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} \gamma_2^{(n)}(u_n).\end{aligned}$$

From the last result and (5) it is obtained that

$$(V_n - E) \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} \gamma_1^{(n)}(u_n) + i(V_n + E) \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} \gamma_2^{(n)}(u_n) = 0, \quad n \geq 1.$$

On the other hand, since $A_n > 0$, $n \geq 1$, then the last relation is equivalent to the following equation

$$\begin{aligned}& \left(\begin{pmatrix} A_n^{-1/2} & 0 \\ 0 & A_n^{-1/2} \end{pmatrix} V_n \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} - E \right) \gamma_1^{(n)}(u_n) + \\ & + i \left(\begin{pmatrix} A_n^{-1/2} & 0 \\ 0 & A_n^{-1/2} \end{pmatrix} V_n \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} - E \right) \gamma_2^{(n)}(u_n) = 0,\end{aligned}$$

$$u_n \in D\left(\text{cl}\left(\text{Re}\widetilde{L_n}\right)\right), \quad n \geq 1.$$

From the last relation, (4) and uniqueness of operator W_n it is clear that

$$W_n = \begin{pmatrix} A_n^{-1/2} & 0 \\ 0 & A_n^{-1/2} \end{pmatrix} V_n \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix},$$

i.e operator

$$V_n = \begin{pmatrix} A_n^{1/2} & 0 \\ 0 & A_n^{1/2} \end{pmatrix} W_n \begin{pmatrix} A_n^{-1/2} & 0 \\ 0 & A_n^{-1/2} \end{pmatrix} : \mathfrak{H}_n \rightarrow \mathfrak{H}_n, \quad n \geq 1$$

must be unitary operator.

Finally, we give one result on the spectrum structure of the normal extensions.

3.3 Theorem. For the spectrum of the normal extension $\widetilde{L} = \bigoplus_{n=1}^{\infty} \widetilde{L}_n$ of the minimal operator L_0 in the space L^2 the following formulas are true

$$\sigma_p(\widetilde{L}) = \bigcup_{n=1}^{\infty} \sigma_p(\widetilde{L}_n), \quad \bigcap_{n=1}^{\infty} \sigma_c(A_n) \subset \sigma_c(A) \subset \bigcup_{n=1}^{\infty} \sigma_c(A_n).$$

3.4 Remark. In this work for the simplicity of explanation the multipoint differential-operator expression has been considered in form (1)-(2). However using the established results in [10] the analogous claims can be obtained in the case when operator coefficient in (1)-(2) are any normal operator in H_n , $n \geq 1$.

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